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Free-fermion branches in some quantum spin models

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Abstract

Extensive numerical analysis of the eigenspectra of the $SU_q(N)$ invariant Perk–Schultz Hamiltonian shows some simple regularities for a significant part of the eigenspectrum. Inspired by those results we have found two sets of solutions of the associated nested Bethe ansatz equations. The first set is obtained at a special value of the anisotropy ($q = \exp(i\pi(N-1)/N)$) and describes, in particular, the ground state and nearby excitations as a sum of free-fermion quasienergies. The second set of solutions provides the energies in the sectors whose number n_i of particles of distinct species ($i = 0, \dots, N-1$) are less than or equal to unity except for one of the species. For this last set we obtain the eigenspectra of a free-fermion model for arbitrary values of the anisotropy.

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1. Introduction

The nested Bethe ansatz is probably the most sophisticated algebraic construction of eigenvectors for integrable lattice models, where the underlying quantum group is of rank larger than one. In the same way as the usual Bethe ansatz, which was widely used for more than 60 years (see, e.g., [1] for reviews), the amplitudes of the wavefunctions are expressed in terms of a sum of plane waves whose wave numbers are given in terms of non-linear and highly non-trivial coupled equations known as the nested Bethe ansatz equations (NBAE).

Except in the thermodynamic limit, in very few cases can the associated NBAE be solved analytically. The exceptions happen for a small number of ‘spin waves’ or for very small lattices. In addition to these, in the case where the equations are of non-nested type, as in the XXZ chain, analytical solutions for chains of arbitrary length can also be derived for the free-fermion point (anisotropy $\Delta = 0$) and for the special anisotropy $\Delta = -\frac{1}{2}$ [2–4].

In a recent paper [5], we presented a new set of analytical solutions of the NBAE for finite quantum chains. These announced solutions correspond to the $SU_q(3)$ Perk–Schultz model [6], or the anisotropic $SU(3)$ Sutherland model [7], with periodic or open boundary conditions, at a special value of the anisotropy. These solutions are given in terms of $p_0(p_1)$ wave numbers of type 1 (type 2). In the case they do not satisfy the usual conditions $p_0 < p_1 < L$, it is not clear, although they are solutions of the NBAE, whether or not they correspond to zero-norm states. Moreover, even in the case where they correspond to physical eigenstates of the quantum Hamiltonian it is not clear, in general, in what eigensector we should expect these solutions. In the case of periodic boundary conditions, by using the functional equations of the model we were able to partially answer the above points in [5].

In the present paper, we fill the above deficiency by presenting a set of new analytical solutions of the NBAE having the number of wave numbers (p_0, p_1) satisfying the usual bounds $p_0 < p_1 < L$, and consequently explaining the free-fermion part of the eigenspectrum of the $SU(3)$ Perk–Schultz model conjectured from extensive numerical calculations in [5]. Moreover, our further ‘experimental’ (i.e. numerical) investigations lead to new observations concerning the anisotropic $SU(N)$ Perk–Schultz quantum chains. Based on our numerical evidence we again have formulated several conjectures. Surprisingly, the inspirations coming from these numerical observations enable us to find analytically a lot of solutions of the related NBAE, which explains part of these new conjectures. The list of analytical solutions we describe in this paper are:

- the sectors $(n_0, n_1, n_2) = (k, k + 1, L - 2k - 1)$ of the $SU(3)$ model with $q = \exp(2i\pi/3)$ with periodic boundary conditions.
- the sectors $(k, k, L - 2k)$ and $(k, k + 1, L - 2k - 1)$ of the $SU(3)$ model with $q = \exp(2i\pi/3)$ with free boundary conditions.
- the sector $(n_0, n_1, \dots, n_{N-1}) = (1, 1, \dots, 1, L - N + 1)$ of the $SU(N)$ model with arbitrary values of q and free boundary conditions.
- the sector $(n_0, n_1, \dots, n_{N-1}) = (1, 1, \dots, 1, 2, L - N)$ of the $SU(N)$ model with $q = \exp((N - 1)i\pi/N)$ and free boundary conditions.
- the sector $(n_0, n_1, n_2, n_3) = (1, 2, 2, L - 5)$ of the $SU(4)$ model with $q = \exp(i3\pi/4)$ and free boundary conditions⁴.

This paper is organized as follows. In section 2 we introduce the model and give the NBAE for periodic and free boundary conditions. In section 3 we consider the set of analytical solutions related to the $SU(3)$ model with $q = \exp(2i\pi/3)$, with periodic and free boundary conditions. In section 4 we consider the NBAE for the sector $(1, \dots, 1, L - N + 1)$, with arbitrary values of the anisotropy. We consider in detail the $SU(4)$ case and formulate a mathematical statement that allows us to generalize our consideration to the case of arbitrary rank. Using a special case of this statement in section 5 we find the NBAE free-fermion solutions for the $SU(N)$ model with anisotropy $q = \exp((N - 1)i\pi/N)$ in the eigensector $(1, \dots, 1, 2, L - N)$. Section 6 contains a set of new ‘experimental’ conjectures that merged from the extensive numerical diagonalizations of the $SU(N)$ quantum chains for arbitrary values of N . Finally, in section 7 we present our conclusions and a summary of our results. We also present in appendices A and B the solutions in the sector $(1, 2, 2, L - 5)$ of the $SU(4)$ model with anisotropy $q = \exp(i3\pi/4)$ and the wavefunctions related to the solutions considered in section 4, respectively.

⁴ We suspect that this class of solutions can be greatly generalized for the $SU(N)$ model with $q = \exp((N - 1)i\pi/N)$ (see discussion in the last section).

2. The $SU(N)$ Perk–Schultz model

The $SU(N)$ Perk–Schultz model [6] is the anisotropic version of the $SU(N)$ Sutherland model [7] with the Hamiltonian, in an L -site chain, given by

$$H_q^p = \sum_{j=1}^{L-1} H_{j,j+1} + p H_{L,1} \quad (p = 0, 1) \quad (1)$$

where

$$H_{i,j} = - \sum_{a=0}^{N-1} \sum_{b=a+1}^{N-1} (E_i^{ab} E_j^{ba} + E_i^{ba} E_j^{ab} - q E_i^{aa} E_j^{bb} - 1/q E_i^{bb} E_j^{aa}).$$

The $N \times N$ matrices E^{ab} have elements $(E^{ab})_{cd} = \delta_c^a \delta_d^b$ and $q = \exp(i\eta)$ plays the role of the anisotropy of the model. The cases of free and periodic boundary conditions are obtained by setting $p = 0$ and $p = 1$ in (1), respectively. This Hamiltonian describes the dynamics of a system containing N classes of particles $(0, 1, \dots, N-1)$ with on-site hard-core exclusion. The number of particles belonging to each species is conserved separately. Consequently, the Hilbert space can be split into block disjoint sectors labelled by $(n_0, n_1, \dots, n_{N-1})$, where $n_i = 0, 1, \dots, L$ is the number of particles of species i ($i = 0, 1, \dots, N-1$). The Hamiltonian (1) has a S_N symmetry due to its invariance under the permutation of distinct particle species, which implies that all the energies can be obtained from the sectors $(n_0, n_1, \dots, n_{N-1})$, where $n_0 \leq n_1 \leq \dots \leq n_{N-1}$ and $n_0 + n_1 + \dots + n_{N-1} = L$.

At $q = 1$ the model is $SU(N)$ invariant and for $q \neq 1$ the model has a $U(1)^{\otimes N-1}$ symmetry as a consequence of the above-mentioned conservation. Moreover, in the special case of free boundaries ($p = 0$), the quantum chain (1) has a larger symmetry, being invariant under the additional quantum $SU(N)_q$ symmetry. This last invariance implies that all the eigenenergies belonging to the sector $(n'_0, n'_1, \dots, n'_{N-1})$ with $n'_0 \leq n'_1 \leq \dots \leq n'_{N-1}$ are degenerate with the energies belonging to the sectors $(n_0, n_1, \dots, n_{N-1})$ with $n_0 \leq n_1 \leq \dots \leq n_{N-1}$, if $n'_0 \leq n_0$ and $n'_0 + n'_1 \leq n_0 + n_1$ and so on up to $n'_0 + n'_1 + \dots + n'_{N-2} \leq n_0 + n_1 + \dots + n_{N-2}$.

The NBAE that give the eigenenergies of the $SU_q(N)$ Perk–Schultz model in the sector whose number of particles is n_i ($i = 0, \dots, N-1$) are given by (see, e.g., [8, 9])

$$\prod_{j=1, j \neq i}^{p_k} F(u_i^{(k)}, u_j^{(k)}) = \prod_{j=1}^{p_{k-1}} f(u_i^{(k)}, u_j^{(k-1)}) \prod_{j=1}^{p_{k+1}} f(u_i^{(k)}, u_j^{(k+1)}) \quad (2)$$

where $k = 0, 1, \dots, N-2$ and $i = 1, 2, \dots, p_k$. The integer parameters p_k are given by

$$p_k = \sum_{i=0}^k n_i \quad k = 0, 1, \dots, N-2 \quad p_{-1} = 0 \quad p_{N-1} = L. \quad (3)$$

The functions $F(x, y)$ and $f(x, y)$ are defined by

$$F(x, y) = \frac{\sin(x-y-\eta)}{\sin(x-y+\eta)} \quad f(x, y) = \frac{\sin(x-y-\eta/2)}{\sin(x-y+\eta/2)} \quad (4)$$

for periodic boundary conditions, and by

$$F(x, y) = \frac{\cos(2y) - \cos(2x-2\eta)}{\cos(2y) - \cos(2x+2\eta)} \quad f(x, y) = \frac{\cos(2y) - \cos(2x-\eta)}{\cos(2y) - \cos(2x+\eta)} \quad (5)$$

for the free boundary case. In using the NBAE (2) we deal with variables of different orders $\{u_j^{(k)}\}$ ($k = 0, \dots, N-2$). The number of variables $u_i^{(k)}$ of order k is equal to p_k . The whole set of NBAE consists of subsets of order k which contain precisely p_k equations ($k = 0, 1, \dots, N-2$). Using NBAE, (2), we should consider $\{u_j^{N-1} = 0, j = 1, \dots, L\}$.

The eigenenergies of the Hamiltonian (1) in the sector $(n_0, n_1, \dots, n_{N-1})$ are given in terms of the roots $\{u_j^{(N-2)}\}$:

$$E = - \sum_{j=1}^{p_{N-2}} \left(-q - \frac{1}{q} + \frac{\sin(u_j - \eta/2)}{\sin(u_j + \eta/2)} + \frac{\sin(u_j + \eta/2)}{\sin(u_j - \eta/2)} \right) \quad (6)$$

where to simplify the notation we write $u_j \equiv u_j^{(N-2)}$ ($j = 1, \dots, p_{N-2}$).

All the solutions of the NBAE (2) which are going to be described in this paper satisfy the additional ‘free-fermion’ conditions (FFC):

$$f^L(u_i, 0) = 1 \quad i = 1, \dots, p_{N-2}. \quad (7)$$

In this case, from (5) and (6) the corresponding eigenenergies of the Hamiltonian (1) for the case of free boundaries are given by

$$E = -2 \sum_{j=1}^{p_{N-2}} \left(-\cos \eta + \cos \frac{\pi k_j}{L} \right) \quad 1 \leq k_j \leq L-1. \quad (8)$$

On the other hand, for periodic boundaries we have found solutions for the $SU(3)$ model with $\eta = 2\pi/3$, and for this case relations (4), (6) and (7) give us

$$E = - \sum_{j=1}^{p_1} \left(1 + 2 \cos \frac{2\pi k_j}{L} \right) \quad 1 \leq k_j \leq L. \quad (9)$$

With the additional set of equations merged from the FFC (7), the number of equations in (2) and (7) exceeds the number of variables by p_{N-2} . At first glance we would expect no chance to satisfy the whole system given by (2) and (7). But, surprisingly, the NBAE possess some hidden symmetry. In the next section, for the sake of simplicity, we consider the simplest case of the $SU(3)$ model with $q = \exp(2i\pi/3)$.

3. Free-fermion spectrum for the $SU(3)$ model with $q = \exp(2i\pi/3)$: NBAE solutions

In [5], based on extensive numerical calculations we conjectured the existence of free-fermion solutions given by (8) and (9) for periodic and free boundary conditions, respectively. In the case of periodic boundaries these solutions happen in the sectors $(n_0, n_1, n_2) = (k, k+1, L-2k-1)$ ($0 \leq k \leq (L-1)/2$) while in the open boundary case they belong to the sectors $(k, k, L-2k)$ or $(k, k+1, L-2k-1)$. In [5], exploring the functional equations related to the NBAE we were able to partially explain these conjectures in the periodic case. In the free boundary case we derived the NBAE directly. These solutions, however, do not belong to the numerically predicted sectors and do not satisfy the usual bounds $p_0 = n_0 < p_1 = n_0 + n_1 < L$. Moreover, it is not clear if they correspond to non-zero-norm eigenfunctions. In this section, we present a direct explanation of these conjectures without the use of functional equations.

The NBAE for the anisotropic $SU(3)$ Perk–Schultz model with anisotropy $q = \exp(2i\pi/3)$, with roots satisfying the FFC, can be written as follows:

$$\prod_{j=1, j \neq i}^{p_0} f(v_i, v_j) \prod_{j=1}^{p_1} f(v_i, u_j) = 1 \quad i = 1, 2, \dots, p_0 \quad (10)$$

$$\prod_{j=1, j \neq i}^{p_1} f(u_i, u_j) \prod_{j=1}^{p_0} f(u_i, v_j) = 1 \quad i = 1, 2, \dots, p_1 \quad (11)$$

$$f^L(u_i, 0) = 1 \quad i = 1, 2, \dots, p_1 \quad (12)$$

where $p_0 = n_0$, $p_1 = n_0 + n_1$ and we have used the relation

$$F(x, y) = 1/f(x, y) \quad (13)$$

which is valid for $q = \exp(\frac{2i\pi}{3})$ or $\eta = \frac{2\pi}{3}$.

3.1. Periodic boundary condition

Before considering the general case let us restrict ourselves initially to the particular eigensector $(1, 2, L - 3)$. We have in this case $p_0 = n_0 = 1$ and $p_1 = n_0 + n_1 = 3$. For this simple case the first subsystem (10) consists of a single equation,

$$f(v, u_1)f(v, u_2)f(v, u_3) = 1. \quad (14)$$

From the definition of the function f given in (4) we can show that the last equation (14) is equivalent to

$$\cos(v + u_1 - u_2 - u_3) + \cos(v - u_1 + u_2 - u_3) + \cos(v - u_1 - u_2 + u_3) = 0. \quad (15)$$

It is clear that this relation has S_3 symmetry under the permutation of the variables u_1, u_2 and u_3 . Surprisingly, it has also S_4 symmetry under the permutation of the variables u_1, u_2, u_3 and v .

The three equations of the second subsystem (11) can be obtained from (10) by the permutations $v \leftrightarrow u_i$, so they also can be reduced to (15). Fixing the variables $u_i, i = 1, 2, 3$, satisfying the FFC (12) and finding v from equation (15) we obtain the solution for the whole system (10)–(12) in the eigensector $(1, 2, L - 3)$.

We now show that this method can be generalized to any sector $(k, k + 1, L - 2k - 1)$, supporting conjecture 1 of our previous paper [5]. Let us consider the set $\{x_i, i = 1, \dots, p_0 + p_1\}$ of variables formed by the union of the two systems of variables:

$$x_i = v_i \quad (i = 1, 2, \dots, p_0) \quad x_{i+p_0} = u_i \quad (i = 1, 2, \dots, p_1). \quad (16)$$

In terms of these variables the system of equations (10)–(12) becomes

$$r(x_i) = f(x_i, x_i) \quad i = 1, 2, \dots, p_0 + p_1 \quad (17)$$

$$f^L(x_i, 0) = 1 \quad i = p_0 + 1, \dots, p_0 + p_1 \quad (18)$$

where $r(x) = \prod_{j=1}^{p_0+p_1} f(x, x_j)$. The first of these subsystems (17) possesses $S_{p_0+p_1}$ permutation symmetry. We now intend to show that many of the equations in this subsystem are dependent, and for $p_0 = k$ and $p_1 = 2k + 1$ we can also satisfy the FFC independently (18). From (4) we may write

$$r(x) = \prod_{j=1}^{p_0+p_1} \frac{\sin(x - x_j - \pi/3)}{\sin(x - x_j + \pi/3)} = \prod_{j=1}^{p_0+p_1} \frac{b - qb_j}{qb - b_j} \quad (19)$$

where, for convenience, we introduced the new variables

$$b = \exp(2ix) \quad b_j = \exp(2ix_j) \quad q = \exp(2i\pi/3). \quad (20)$$

The first subsystem (17) now becomes a system of algebraic equations:

$$\prod_{j=1}^{p_0+p_1} (b_i - qb_j) + \prod_{j=1}^{p_0+p_1} (qb_i - b_j) = 0 \quad i = 1, 2, \dots, p_0 + p_1. \quad (21)$$

Using the standard symmetric functions

$$\begin{aligned}
 S_0 &= 1 \\
 S_1 &= b_1 + b_2 + \cdots + b_{p_0+p_1} \\
 S_2 &= b_1 b_2 + b_1 b_3 + \cdots + b_{p_0+p_1-1} b_{p_0+p_1} \\
 &\dots \\
 S_m &= \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq p_0+p_1} b_{i_1} b_{i_2} \cdots b_{i_m} \\
 &\dots \\
 S_{p_0+p_1} &= b_1 b_2 \cdots b_{p_0+p_1}
 \end{aligned} \tag{22}$$

we can rewrite (21) as

$$\sum_{m=0}^{p_0+p_1} (-1)^m S_m b_i^{p_0+p_1-m} (q^m + q^{p_0+p_1-m}) = 0. \tag{23}$$

Adding this last equation to the identity

$$q^{-p_0-p_1} \prod_{j=1}^{p_0+p_1} (b_i - b_j) = \sum_{m=0}^{p_0+p_1} (-1)^m S_m b_i^{p_0+p_1-m} q^{-p_0-p_1} = 0 \tag{24}$$

we obtain

$$\sum_{m=0}^{p_0+p_1} (-1)^m S_m b_i^{p_0+p_1-m} (q^m + q^{p_0+p_1-m} + q^{-p_0-p_1}) = 0. \tag{25}$$

For $q = \exp(2i\pi/3)$ we have the following possibilities:

$$\begin{aligned}
 q^m + q^{p_0+p_1-m} + q^{-p_0-p_1} &= 3q^{-p_0-p_1} && \text{for } p_0 + p_1 + m = 3n \\
 q^m + q^{p_0+p_1-m} + q^{-p_0-p_1} &= 0 && \text{for } p_0 + p_1 + m \neq 3n.
 \end{aligned} \tag{26}$$

Let $p_0 + p_1 = 3k + r$, where k is an integer and $r \in \{-1, 0, 1\}$. Inserting (26) into (25) we obtain

$$\begin{aligned}
 \sum_{\mu=0}^k (-1)^\mu S_{3\mu} b_i^{-3\mu} &= 0 && \text{for } r = 0 \\
 \sum_{\mu=0}^{k-1} (-1)^\mu S_{3\mu+2} b_i^{-3\mu} &= 0 && \text{for } r = 1 \\
 \sum_{\mu=0}^{k-1} (-1)^\mu S_{3\mu+1} b_i^{-3\mu} &= 0 && \text{for } r = -1
 \end{aligned} \tag{27}$$

where $i = 1, 2, \dots, p_0 + p_1$. We then see that the subsystem (17) can be satisfied if we impose

$$S_{3v+\rho} = 0 \quad v = 0, 1, \dots, k-1 \tag{28}$$

where $\rho = 0$ for $r = 0$, $\rho = 2$ for $r = 1$ and $\rho = 1$ for $r = -1$. Since $S_0 = 1$ it is not possible to obtain solutions of this type for $L = 3k$ ($\rho = 0$) and we have to limit ourselves to the cases where $p_0 + p_1 \neq 3k$.

Let us now fix p_1 variables u_i , satisfying the FFC (18). Variables $v_i, i = 1, 2, \dots, p_0$, can in principle be found from system (28). In order to do that we use the decomposition of

the symmetric functions depending on two sets of variables, namely,

$$\begin{aligned}
 S_0 &= 1 \\
 S_1 &= s_1 + \sigma_1 \\
 S_2 &= s_2 + s_1\sigma_1 + \sigma_2 \\
 &\dots \\
 S_m &= \sum_{k=\max\{0, m-p_0\}}^{\min\{m, p_1\}} s_{m-k}\sigma_k \\
 &\dots \\
 S_{p_0+p_1} &= s_{p_0}\sigma_{p_1}
 \end{aligned} \tag{29}$$

where σ_i are the symmetric combinations of the known variables u_i and s_i are the symmetric combinations of the unknown variables v_j . Consequently, system (28) can be reduced to a linear system for symmetric functions $s_i, i = 1, 2, \dots, p_0$. This system can be solved if the number of variables p_0 is greater than or equal to the number of equations k . Then we have the system

$$p_0 = n_0 \quad p_1 = n_0 + n_1 \quad n_0 \leq n_1 \quad p_0 + p_1 = 3k \pm 1 \quad k \leq p_0. \tag{30}$$

These relations give us the constraints $n_0 = k$ and $n_1 = k + 1$, and consequently $p_0 + p_1 = 2n_0 + n_1 = 3k + 1$, implying that solutions of (28) exist only for $\rho = 2$. These solutions happen in the sector $(k, k + 1, L - 2k - 1)$. Consider for illustration the case $k = 2$ ($p_0 = 2$ and $p_1 = 5$). We have five variables u_1, \dots, u_5 , which we fix with the FFC (18), and two unknown variables v_1, v_2 . Using (29) we can write system (28) as follows:

$$S_2 = s_2 + s_1\sigma_1 + \sigma_2 = 0 \quad S_5 = s_2\sigma_3 + s_1\sigma_4 + \sigma_5 = 0. \tag{31}$$

We know the functions $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and σ_5 , so we have two linear equations for $s_1 = v_1 + v_2$ and $s_2 = v_1 v_2$.

3.2. Free boundary condition

The case of free boundary conditions is slightly more complicated. Inserting definition (5) into subsystem (17) we obtain the system

$$\begin{aligned}
 \sin(2x_i - 2\pi/3) \prod_{j=1}^{p_0+p_1} (\cos(2x_j) - \cos(2x_i - 2\pi/3)) + \sin(2x_i + 2\pi/3) \prod_{j=1}^{p_0+p_1} (\cos(2x_j) \\
 - \cos(2x_i + 2\pi/3)) = 0 \quad i = 1, 2, \dots, p_0 + p_1.
 \end{aligned} \tag{32}$$

As in the periodic case we may use the symmetric functions

$$\begin{aligned}
 S_0 &= 1 \\
 S_1 &= \cos 2x_1 + \cos 2x_2 + \dots + \cos 2x_{p_0+p_1} \\
 &\dots \\
 S_{p_0+p_1} &= \cos 2x_1 \cos 2x_2 \dots \cos 2x_{p_0+p_1}
 \end{aligned} \tag{33}$$

to rewrite (32) as

$$\begin{aligned}
 \sum_{m=0}^{p_0+p_1} S_{p_0+p_1-m} (\sin(2x_i - 2\pi/3) \cos^m(2x_i - 2\pi/3) \\
 + \sin(2x_i + 2\pi/3) \cos^m(2x_i + 2\pi/3)) = 0 \quad i = 1, 2, \dots, p_0 + p_1.
 \end{aligned} \tag{34}$$

On the other hand we have the identities

$$\begin{aligned} \sin 2x_i \prod_{j=1}^{p_0+p_1} (\cos 2x_j - \cos 2x_i) &= 0 \\ &= \sum_{m=0}^{p_0+p_1} S_{p_0+p_1-m} \sin 2x_i \cos^m 2x_i \quad i = 1, 2, \dots, p_0 + p_1. \end{aligned} \quad (35)$$

Adding equations (35) and (36) we obtain

$$\sum_{m=0}^{p_0+p_1} S_{p_0+p_1-m} \phi_m(x_i) = 0 \quad i = 1, 2, \dots, p_0 + p_1 \quad (36)$$

where the functions $\phi_m(x)$ are given by

$$\phi_m(x) = \sum_{\rho=-1}^1 \sin(2x + 2\pi\rho/3) \cos^m(2x + 2\pi\rho/3). \quad (37)$$

It is now convenient to find a recursion relation for these functions. In order to do that let us consider a general sequence

$$y_m = a\alpha^m + b\beta^m + c\gamma^m. \quad (38)$$

This sequence satisfies the recurrence relation

$$y_{m+1} + Ay_m + By_{m-1} + Cy_{m-2} = 0 \quad (39)$$

if A , B and C are the coefficients of the third degree equation

$$t^3 + At^2 + Bt + C \equiv (t - \alpha)(t - \beta)(t - \gamma) \quad (40)$$

with roots α , β and γ . Consequently, our functions $\phi_m(x)$ satisfy the recurrence relation

$$\phi_{m+1}(x) + A\phi_m(x) + B\phi_{m-1}(x) + C\phi_{m-2}(x) = 0 \quad (41)$$

where

$$t^3 + At^2 + Bt + C \equiv \prod_{\rho=-1}^1 (t - \cos(2x + 2\pi\rho/3)) = t^3 - \frac{3}{4}t - \frac{1}{4}\cos 6x. \quad (42)$$

Due to (39) one then has the recurrence relation for the functions ϕ_m :

$$\phi_{m+1}(x) = \frac{3}{4}\phi_{m-1}(x) + \frac{1}{4}\cos 6x \phi_{m-2}(x). \quad (43)$$

In order to iterate (43) we also need ϕ_0 , ϕ_1 and ϕ_2 . Using definitions (37) we find

$$\phi_0(x) = 0 \quad \phi_1(x) = 0 \quad \text{and} \quad \phi_2(x) = \frac{3}{4}\sin 6x. \quad (44)$$

Let us give a list of several $\phi_i(x)$, which we obtain by using the initial functions (44) and the recurrence relations (43):

$$\begin{aligned} \phi_3(x) &= 0 & \phi_4(x) &= \frac{9}{16}\sin 6x & \phi_5(x) &= \frac{3}{16}\sin 6x \cos 6x \\ \phi_6(x) &= \frac{27}{64}\sin 6x & \phi_7(x) &= \frac{9}{32}\sin 6x \cos 6x \\ \phi_8(x) &= \sin 6x \left(\frac{81}{256} + \frac{3}{64}\cos^2 6x \right). \end{aligned} \quad (45)$$

It is clear from (44) and (43) that $\phi_m(x) = \sin 6x \Phi_m(\cos 6x)$, where $\Phi_m(t)$ is a polynomial. The degree μ of this polynomial depends upon the number m . One can prove by finite induction that for $m = 3n + r$, $r = 1, 2, 3$, $\phi_m = 0$ if $n - 2 + r < 0$ and otherwise it is a polynomial of degree $\mu = n - 2 + r$.

Returning to our main equation (36) and using the value we just obtained for μ we easily verify that the left-hand side of this equation is (up to the common factor $\sin 6x$) a polynomial of $\cos 6x$ with degree $\nu = [(p_0 + p_1 - 2)/3]$, where $[x]$ means the integer part of x . We can try to convert equation (36) into an identity on the variable x by equating to 0 all the coefficients of the polynomial, i.e.,

$$\sum_{m=0}^{p_0+p_1} S_{p_0+p_1-m} \Phi_m(\cos 6x) = 0. \quad (46)$$

We obtain $k = \nu + 1 = [(p_0 + p_1 + 1)/3]$ linear equations on the symmetric combinations S_m . Before considering the general case let us consider, to begin with, the cases where $p_0 + p_1 = 6, 7$ and 8 . Using (45), the relation between the functions ϕ and Φ and equation (46), we obtain

$$S_4 + \frac{3}{4} S_2 + \frac{1}{4} S_1 \cos 6x + \frac{9}{16} = 0 \quad (p_0 + p_1 = 6) \quad (47)$$

$$S_5 + \frac{3}{4} S_3 + \frac{1}{4} S_2 \cos 6x + \frac{9}{16} S_1 + \frac{3}{8} \cos 6x = 0 \quad (p_0 + p_1 = 7) \quad (48)$$

$$S_6 + \frac{3}{4} S_4 + \frac{1}{4} S_3 \cos 6x + \frac{9}{16} S_2 + \frac{3}{8} S_1 \cos 6x + \frac{27}{64} + \frac{1}{16} \cos^2 6x = 0 \quad (p_0 + p_1 = 8). \quad (49)$$

We see that for $p_0 + p_1 = 6$ and for $p_0 + p_1 = 7$ one has two equations

$$S_1 = 0 \quad S_4 + \frac{3}{4} S_2 + \frac{9}{16} = 0 \quad (50)$$

and

$$S_2 + \frac{3}{2} = 0 \quad S_5 + \frac{3}{4} S_3 + \frac{9}{16} S_1 = 0 \quad (51)$$

respectively. As far as $p_0 + p_1 = 8$ is concerned, the last term $\frac{1}{16} \cos^2 6x$ makes it impossible to convert (46) into an identity.

Let us consider the general case. In the same way as in the periodic boundary case we fix the variables u_i ($i = 1, \dots, p_1$) satisfying the FFC (18) and try to find the variables v_i ($i = 1, \dots, p_0$) using $k = [(p_0 + p_1 + 1)/3]$ equations, which we obtain by imposing that all the coefficients of polynomial (46) are zero. In order to find a solution the number of equations k cannot exceed the number of variables p_0 . All the restrictions

$$p_0 = n_0 \quad p_1 = n_0 + n_1 \quad n_0 \leq n_1 \quad p_0 \geq [(p_0 + p_1 + 1)/3] \quad (52)$$

reduced to two variants: $n_0 = n_1 = k$, $p_0 + p_1 = 3k$ and $n_0 = k$, $n_1 = k + 1$, $p_0 + p_1 = 3k + 1$, which correspond to the sectors $(k, k, L - 2k)$ and $(k, k + 1, L - 2k - 1)$, respectively. One can show using the recurrence relation (43) and the initial functions (44) that as long as $p_0 + p_1 \neq 3k + 2$ we can obtain a consistent linear system for the symmetric functions S_i . We use these linear equations in the same way as we have used (28) in the periodic case and this explains conjecture 3 of [5]. For example, systems (50) and (51) give us free-fermion solutions for the sectors $(2, 2, L - 4)$ and $(2, 3, L - 5)$.

4. Free-fermion solutions of the NBAE for generic q

In this section, differently from the results of the last section, where the free-fermion solutions were found for a specific value of q , we are going to present free-fermion solutions of the NBAE that are valid for arbitrary values of q . These solutions are valid only for free boundary conditions and will happen in the special sectors of the $SU(N)_q$ model where the number of particles of each distinct species is at most one, except for one of the species, which can

be considered as the background (holes, for example). The $S(U)_q$ symmetry ensures that the general sector containing all these solutions is the special sector where $(n_0, \dots, n_{N-1}) = (1, \dots, 1, L - N + 1)$, which gives from definition (3) the values

$$p_0 = 1, p_1 = 2, \dots, p_{N-2} = N - 1, p_{N-1} = L. \quad (53)$$

For this case the NBAE (2) consist of $N - 1$ subsets ($k = 0, 1, \dots, N - 1$) of equations where the k th subset has precisely $k + 1$ equations.

To illustrate our procedure let us consider, for simplicity, the $SU(4)$ model in the sector $(1, 1, 1, L - 3)$. In this case, we have three subsets ($k = 0, 1, 2$) and three groups of variables $(u^{(0)}, u^{(1)}, u^{(2)})$:

$$1 = f(w, v_1) f(w, v_2) \quad (54)$$

$$F(v_1, v_2) = f(v_1, w) \times f(v_1, u_1) f(v_1, u_2) f(v_1, u_3) \quad (55)$$

$$F(v_2, v_1) = f(v_2, w) \times f(v_2, u_1) f(v_2, u_2) f(v_2, u_3)$$

$$F(u_1, u_2) F(u_1, u_3) = f(u_1, v_1) f(u_1, v_2) \quad (56)$$

$$F(u_2, u_1) F(u_2, u_3) = f(u_2, v_1) f(u_2, v_2)$$

$$F(u_3, u_1) F(u_3, u_2) = f(u_3, v_1) f(u_3, v_2)$$

where we write w instead of $u^{(0)}$, v_i instead of $u_i^{(1)}$ ($i = 1, 2$) and u_i instead of $u_i^{(2)}$ ($i = 1, 2, 3$). By using the FFC (7), the third subset can be written in the simplified form (56). Inserting (5) into the first equation (54) we obtain promptly two possibilities

$$\sin 2w = 0 \quad (57)$$

$$\cos 2v_1 + \cos 2v_2 - 2 \cos \eta \cos 2w = 0. \quad (58)$$

We do not intend to consider for the moment the first possibility (57). Imposing condition (58) we can check (eliminating, for example, the variable w) that for (v_1, v_2, w) satisfying (58), besides (54) we also have two additional equalities, namely,

$$F(v_1, v_2) = f(v_1, w) \quad F(v_2, v_1) = f(v_2, w). \quad (59)$$

Consequently, the second subsystem becomes

$$1 = f(v_1, u_1) f(v_1, u_2) f(v_1, u_3) \quad 1 = f(v_2, u_1) f(v_2, u_2) f(v_2, u_3). \quad (60)$$

We will show below that the set of five equations formed by the second (60) and the third subsets (56) contains only two independent equations, and consequently we may fix the variables u_i ($i = 1, 2, 3$) by imposing the FFC (17) and find the two variables v_1 and v_2 from these equations. The remaining variable w is then obtained from (58). Consequently, we find the free-fermion eigenspectra (8) for arbitrary values of q or η .

The previous analysis can be extended for the sector $(1, \dots, 1, L - N + 1)$ for general $SU(N)_q$ by exploring the general theorem:

For any k fixed numbers u_1, u_2, \dots, u_k one can find $k - 1$ numbers v_1, v_2, \dots, v_{k-1} satisfying the two sets of equations

$$\prod_{j=1}^k f(v_i, u_j) = 1 \quad (i = 1, 2, \dots, k - 1) \quad (61)$$

$$\prod_{j=1, j \neq i}^k F(u_i, u_j) = \prod_{j=1}^{k-1} f(u_i, v_j) \quad (i = 1, 2, \dots, k). \quad (62)$$

Leaving the proof of the above theorem for the moment, we can see that applying this theorem to $(k - 1)$ known numbers v_1, \dots, v_{k-1} , one can find the numbers w_1, \dots, w_{k-2}

satisfying the equations

$$\prod_{j=1}^{k-1} f(w_i, v_j) = 1 \quad (i = 1, 2, \dots, k-2) \quad (63)$$

$$\prod_{j=1, j \neq i}^{k-1} F(v_i, v_j) = \prod_{j=1}^{k-2} f(v_i, w_j) \quad (i = 1, 2, \dots, k-1). \quad (64)$$

Multiplying the i th ($i = 1, \dots, k-1$) equation of sets (61) and (64) we obtain literally one of the subsets of the NBAE:⁵

$$\prod_{j=1, j \neq i}^{k-1} F(v_i, v_j) = \prod_{j=1}^{k-2} f(v_i, w_j) \prod_{j=1}^k f(v_i, u_j) \quad (i = 1, 2, \dots, k). \quad (65)$$

Applying the above theorem $k-1$ times we obtain a tower of numbers:

$$\begin{aligned} &u_1, u_2, u_3, \dots, u_{k-2}, u_{k-1}, u_k \\ &v_1, v_2, \dots, v_{k-2}, v_{k-1} \\ &w_1, \dots, w_{k-2} \\ &\dots \\ &\dots \\ &y_1, y_2 \\ &z. \end{aligned}$$

This implies that if we begin by fixing the $k = N-1$ variables u_i ($i = 1, \dots, k$), satisfying the FFC (17) we obtain by using construction (65) recursively the solution of the NBAE of the $SU(N)$ Perk–Schultz model, with free boundaries, in the sector $(n_0, n_1, \dots, n_{N-1}) = (1, 1, \dots, 1, L-N+1)$. The free-fermion-like energies are given by (8) for arbitrary values of η . The previous results (56) and (60) are just consequences of the particular case where $k = 3$.

Let us now prove the announced theorem. Let us fix $\{u_j\}$, $j = 1, 2, \dots, k$. Equation (61) can then be written as follows:

$$P(v_i) = 0 \quad (i = 1, 2, \dots, k) \quad (66)$$

where

$$P(v) \equiv \prod_{j=1}^k (\cos 2u_j - \cos(2v - \eta)) - \prod_{j=1}^k (\cos 2u_j - \cos(2v + \eta)). \quad (67)$$

The use of the symmetric functions (33) (u_j instead of x_i) allows us to write

$$P(v) \equiv \sum_{m=1}^k (-1)^{m+1} S_{k-m} (\cos^m(2v - \eta) - \cos^m(2v + \eta)). \quad (68)$$

Since $\cos(2v - \eta) - \cos(2v + \eta) = 2 \sin \eta \sin 2v$ and $a^m - b^m = (a - b)(a^{m-1} + a^{m-2}b + \dots + b^{m-1})$ we have

$$P(v) \equiv \sin 2v p(\cos 2v) \quad (69)$$

where $p(t)$ is a polynomial of degree $k-1$. We can factorize this polynomial, and apart from a multiplicative constant (Ω) we can write

$$P(v) = \Omega \sin 2v \prod_{i=1}^{k-1} (\cos 2v - \cos 2b_i). \quad (70)$$

Now (66) is easily solved: $v_i = b_i$, $i = 1, 2, \dots, k-1$.

⁵ In the case $k = N-1$ the FFC (7) is used.

Consider now the right-hand side of the second equation of (62). Relation (70) allows us to write

$$\begin{aligned} \prod_{i=1}^{k-1} f(u_j, v_i) &= \prod_{i=1}^{k-1} f(u_j, b_i) = \prod_{i=1}^{k-1} \frac{\cos(2u_i - \eta) - \cos 2b_i}{\cos(2u_i + \eta) - \cos 2b_i} \\ &= \frac{P(2u_j - \eta) \sin(2u_j + \eta)}{\sin(2u_j - \eta) P(2u_j + \eta)}. \end{aligned} \quad (71)$$

Using expression (67) for $P(v)$ we obtain

$$\prod_{i=1}^{k-1} f(u_j, v_i) = \frac{\sin(2u_j + \eta) \prod_{i=1}^k (\cos 2u_i - \cos(2u_j - 2\eta)) - \prod_{i=1}^k (\cos 2u_i - \cos 2u_j)}{\sin(2u_j - \eta) \prod_{i=1}^k (\cos 2u_i - \cos 2u_j) - \prod_{i=1}^k (\cos 2u_i - \cos(2u_j + 2\eta))}. \quad (72)$$

Since $\prod_{i=1}^k (\cos 2u_i - \cos 2u_j) = 0$, we obtain

$$\prod_{i=1}^{k-1} f(u_j, v_i) = -\frac{\sin(2u_j + \eta)}{\sin(2u_j - \eta)} \prod_{i=1}^k \frac{\cos 2u_i - \cos(2u_j - 2\eta)}{\cos 2u_i - \cos(2u_j + 2\eta)}. \quad (73)$$

One can also check that

$$\frac{\sin(2u_j + \eta)}{\sin(2u_j - \eta)} = -\frac{\cos 2u_j - \cos(2u_j + 2\eta)}{\cos 2u_j - \cos(2u_j - 2\eta)} \quad (74)$$

so that (73) can be written as

$$\prod_{i=1}^{k-1} f(u_j, v_i) = \prod_{i=1, i \neq j}^k F(u_j, u_i) \quad (75)$$

concluding the proof of the theorem.

In appendix B, we derived the wavefunctions corresponding to the free-fermion solutions obtained in this section.

5. Free-fermion spectrum for special values of q . NBAE solutions for the sector $(\mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{2}, L - N)$

For this sector the NBAE of Hamiltonian (1) with free boundaries have about the same system of equations as in the previous section but the number of variables u_i is now $k + 1$ instead of k . We have, in this case, to use a modified theorem:

For any $k+1$ fixed numbers $u_1, u_2, \dots, u_k, u_{k+1}$ one can find $k-1$ numbers v_1, v_2, \dots, v_{k-1} so that the two systems of equations

$$\prod_{j=1}^{k+1} f(v_i, u_j) = 1 \quad (i = 1, 2, \dots, k-1) \quad (76)$$

$$\prod_{j=1, j \neq i}^{k+1} F(u_i, u_j) = \prod_{j=1}^{k-1} f(u_i, v_j) \quad (i = 1, 2, \dots, k) \quad (77)$$

are satisfied for the special anisotropy $\eta = k\pi/(k+1)$.

We begin the proof as in the case of the previous theorem and as in (69) the corresponding polynomial $P(v)$ we obtain would have degree k ; however, for the special anisotropy value $\eta = k\pi/(k+1)$ we can show that the degree of $P(v)$ decreases to $k-1$, and we have the same situation as in the previous section. Consequently, the proof follows straightforwardly in the same way as before and we have for the sector $(1, \dots, 1, 2, L-N)$ the free-fermion energies given by (8), with $p_{N-2} = N$ and $\eta = (N-1)\pi/N$.

6. Free-fermion spectrum for the $SU(N)$ model. Conjectures merged from numerical studies

In this section, we report some conjectures merged from extensive brute force numerical diagonalizations of the quantum chain (1) with free boundary condition. The analytical results presented in the previous sections explain part of these numerical observations. In order to announce these conjectures let us define the special sectors

$$S_k = \left(\left[\frac{k}{N-1} \right], \left[\frac{k+1}{N-1} \right], \dots, \left[\frac{k+N-2}{N-1} \right], L-k \right) \quad k = 0, 1, \dots, L \quad (78)$$

where, as before, $[x]$ means the integer part of x . For example, for $N = 4$ and $L = 7$ the sectors are

$$\begin{aligned} S_0 &= (0, 0, 0, 7) & S_1 &= (0, 0, 1, 6) & S_2 &= (0, 1, 1, 5) \\ S_3 &= (1, 1, 1, 4) & S_4 &= (1, 1, 2, 3) & S_5 &= (1, 2, 2, 2) \\ S_6 &= (2, 2, 2, 1) & S_7 &= (2, 2, 3, 0) \end{aligned} \quad (79)$$

and due to the quantum symmetry of the Hamiltonian we have the special ordering

$$S_0 \subset S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5 \equiv S_6 \supset S_7. \quad (80)$$

This means that, for example, all eigenvalues found in the sector S_3 can also be found in the sectors S_4, S_5 and S_6 , and all eigenvalues appearing in the sector S_7 can also be found in the sectors S_6 and S_5 . Sectors S_5 and S_6 are totally equivalent. In this example, let us call the sectors $S_0, S_1, S_2, S_3, S_4, S_5$ the *left* sectors and S_6, S_7 the *right* ones.

We can generalize this definition to any $L = Nn+r$, where n and r are the natural numbers and $r < N$, obtaining $L-n$ left sectors and $n+1$ right ones. Now we can formulate the conjectures merged from our brute force diagonalizations.

Conjecture 1. For $L = Nn+r$ ($r = 1, 2, \dots, N-1$) the Hamiltonian (1) with $p = 0$ and $q = \exp(i\pi/N)$ has eigenvalues given by

$$E_I = -2 \sum_{j \in I} \left(\cos\left(\frac{\pi}{N}\right) + \cos\left(\frac{j\pi}{L}\right) \right) \quad (81)$$

where I is an arbitrary subset of the set $\{1, 2, \dots, L-1\}$. If k is the number of elements of the subset I and also S_k is a left sector then we find the eigenvalues (81) in the sectors $S_k, S_{k+1}, \dots, S_{L-n}$. On the other hand, if S_k is a right sector then we find the eigenvalues (81) in the sectors $S_{L-n-1}, S_{L-n}, \dots, S_{k+1}$.

For $L = Nn$ we have a slightly more delicate picture. In this case we have the left sectors, the right sectors and a central one (n, n, n, n) .

Conjecture 1'. For $L = Nn$ we can use conjecture 1 considering the central sector as a left or a right one.

This is possible due to the coincidence, apart from degeneracies, of the eigenenergies in the eigensectors (n, n, n, \dots, n) and $(n-1, n, n, \dots, n+1)$.

Consider now the special subsets $I = \{1, 2, 3, \dots, k\}$, $k = 0, 1, \dots, L-1$. Due to conjectures 1 and 1' the Hamiltonian (1) has the corresponding eigenvalues

$$E^{(k)} = -2 \sum_{j=1}^k \left(\cos\left(\frac{\pi}{N}\right) + \cos\left(\frac{j\pi}{L}\right) \right) = 1 - 2k \cos\left(\frac{\pi}{N}\right) - \frac{\sin \pi(2k+1)/2L}{\sin \pi/2L}. \quad (82)$$

We can now formulate the remarkable conjecture:

Conjecture 2. *The eigenvalues (82) are the lowest eigenenergies in the special sectors (78) of the Hamiltonian (1) with anisotropy $q = \exp(i(N-1)\pi/N)$. Namely, for the left sectors we have $E_{\min}(S_k) = E^{(k)}$ while for the right sectors $E_{\min}(S_k) = E^{(k-1)}$.*

We can add here an additional conjecture:

Conjecture 2'. *The eigenvalues (82) for $k = L-n-1$ give the ground-state energy of the Hamiltonian:*

$$E_0 = 1 + 2(1-L+n) \cos\left(\frac{\pi}{N}\right) - \frac{\sin \pi(2n+1)/2L}{\sin \pi/2L}. \quad (83)$$

The last of the special sectors is

$$S_L = \left(\left[\frac{L}{N-1} \right], \left[\frac{L+1}{N-1} \right], \dots, \left[\frac{L+N-2}{N-1} \right], 0 \right). \quad (84)$$

In this sector we have only $N-1$ classes of particles and the Hamiltonian (1) is reduced effectively to a $SU_q(N-1)$ -invariant quantum spin model. The sector is of right type and due to conjecture 2 we can state the following conjecture:

Conjecture 3. *The eigenvalue (82) for $k = L-1$ gives the ground-state energy of the $SU_q(N-1)$ Hamiltonian with the anisotropy $q = \exp(i(N-1)\pi/N)$. This eigenvalue can be written as*

$$E_0 = -2(L-1) \cos\left(\frac{\pi}{N}\right). \quad (85)$$

For $N = 3$, for example, we get the XXZ model with the anisotropy $\Delta = -1/2$ and $E_0 = 1 - L$, a result that was first observed in [2] and produced quite interesting consequences [3, 4].

7. Summary and conclusions

Most of the researches related to exact integrable systems end with the derivation of the Bethe ansatz equations, whose solutions provide, in principle, the eigenvalues and eigenvectors of the Hamiltonian or transfer matrix of the associated model. These equations are highly non-linear and in several cases, thanks to some appropriate guessing on the topology of the roots in the complex plane, they are transformed in the bulk limit ($L \rightarrow \infty$) into integral equations, which provide most of the thermodynamic physical quantities, whose calculations depend upon eigenvalues only. Even in this limit these simplifications do not help in the calculation of quantities depending directly on the eigenvectors such as form factors and correlation functions, since the amplitudes of the eigenvectors, obtained in the Bethe basis, are given as

combinatorial sums of plane waves that are highly non-trivial to operate in the thermodynamic limit.

Since the exact integrability is a lattice size-independent property, the exact solution of the associated Bethe ansatz equations is a quite important step towards the complete mathematical and physical understanding of these models. Due to the complexity of these equations, the solutions are known analytically in very few particular cases, such as the free-fermion case and the XXZ chain with anisotropy $\Delta = -1/2$ [2]. It is interesting to mention that in this last case only some ratios among the amplitudes of the eigenvectors are known [3] and these results provide a quite interesting connection of this problem with that of the sign-alternating matrices [4]. In a previous paper [5] we presented some new solutions of the associated Bethe ansatz equations of the simplest generalization of the XXZ chain with two conserved global quantities, namely the $SU(3)_q$ Perk–Schultz model with periodic and free boundary conditions. In this case, the Bethe ansatz equations are of nested type (NBAE) and these solutions were obtained for the special anisotropy value $q = \exp(i2\pi/3)$, being the first example of analytical solutions of NBAE for finite lattices. The results in [5] were derived by exploring the functional relations merged from the associated NBAE, and presented two difficulties: the solutions for the open lattice happen in sectors out of the usual bounds ($n_0 < n_1 < n_2$), it being unclear whether they correspond to zero-norm states, and the precise sector where the periodic solutions happen is not known. In section 3 of the present paper, in a different way, by adding to the set of NBAE the FFC (12) we were able to solve these difficulties by showing the existence of free-fermion solutions in the sectors satisfying the usual bounds ($n_0 < n_1 < n_2$) for the periodic and open cases. In contrast to the functional method used in [5], the success of the direct method presented in section 3 for the $SU(3)_q$ case enables us to generalize these solutions for some sectors of the Perk–Schultz $SU(N)_q$ models, revealing the existence of free-fermion branches, at the special anisotropy $q = \exp(i2\pi(N-1)/N)$. It is important to mention that our numerical calculations reveal other energies beyond those predicted in the calculated free-fermion branches. Although we did not prove it analytically, an extensive numerical analysis shows that the low-lying energies, including the ground-state energy, belong to those classes of free-fermion-like solutions we have found. In fact, it was these numerical results that induced our analytical calculations. The numerical analysis indicated a set of conjectures presented in section 6, part of them were proved in sections 3–5 and appendix A.

After the success of finding free-fermion solutions at special values of anisotropies of the $SU(N)_q$ model, we tried to find other solutions for other anisotropies, at least for some simple sectors. Surprisingly, we were able to find free-fermion solutions for all values of the anisotropy in the sector where we have at most one particle of each species, except for one of the species (the background one). In section 4, we show the existence of these solutions for the most general sector within this class, i.e. $(n_0, \dots, n_{N-1}) = (1, \dots, 1, L - N + 1)$. Beyond the free-fermion energies degenerate with the lower sectors, in this sector the energies given by the sum of $p_{N-2} = N - 1$ free-fermion energies ϵ_j appear, i.e.,

$$E = \sum_{j=1}^{N-1} \epsilon_j = \sum_{j=1}^{N-1} \left(q + \frac{1}{q} - x_j - \frac{1}{x_j} \right) \quad (86)$$

where

$$x_j = \exp\left(i\frac{\pi k_j}{L}\right) \quad 1 \leq k_j \leq L - 1. \quad (87)$$

These eigenvalue expressions, valid for arbitrary values of q , allow us to expect that a direct analytical derivation of the related eigenfunctions, without the use of the NBAE, might be

possible. In fact, in appendix B we derive these eigenvectors, which are given by combinations of Slater fermionic determinants of one-particle wavefunctions $\Psi_j(m_l)$. Denoting the configurations where the l th particle is at position m_l ($l = 1, \dots, N-1$; $m_l = 1, \dots, L$) as $|m_1, \dots, m_{N-1}\rangle$ the eigenfunctions are given by

$$|\psi_{\{x_1, \dots, x_{N-1}\}}\rangle = \sum_{m_1, \dots, m_{N-1}=1}^L q^{-f(m_1, \dots, m_{N-1})} \times \det \begin{vmatrix} \Psi_1(m_1) & \Psi_1(m_2) & \cdots & \Psi_1(m_{N-1}) \\ \Psi_2(m_1) & \Psi_2(m_2) & \cdots & \Psi_2(m_{N-1}) \\ \cdots & \cdots & \cdots & \cdots \\ \Psi_{N-1}(m_1) & \Psi_{N-1}(m_2) & \cdots & \Psi_{N-1}(m_{N-1}) \end{vmatrix} |m_1, \dots, m_{N-1}\rangle$$

where

$$\Psi_j(m) = \left(1 - \frac{q}{x_j}\right) x_j^m - (1 - qx_j)/x_j^m. \quad (88)$$

The factor $f(m_1, \dots, m_{N-1})$ is an integer given by the minimum number of pair permutations of the set $(m_1, \dots, m_{N-1}) \rightarrow (m'_1, \dots, m'_{N-1})$, such that $m'_1 < m'_2 < \cdots < m'_{N-1}$. It is interesting to observe that at the isotropic point the wavefunction amplitudes are just given by standard free-fermion Slater determinants.

In section 5, continuing our search for free-fermion-like solutions we were able again to find solutions in the sector $(n_0, \dots, n_{N-1}) = (1, \dots, 1, 2, L-N)$. In this case the free-fermion solutions happen only at the special value of the anisotropy $\eta = (N-1)\pi/N$. Although we believe that it should be possible to obtain free-fermion-like solutions for arbitrary eigensectors, this procedure is not straightforward, and each case deserves some ingeniousness. As an another example we show in appendix A the existence of free-fermion solutions in the sector $(n_0, n_1, n_2, n_3) = (1, 2, 2, L-5)$ of the $SU(4)_q$ model at the anisotropy $q = \exp(i3\pi/4)$.

Although we have only derived wavefunctions for the sector $(n_0, \dots, n_{N-1}) = (1, \dots, L-N+1)$, we expect to obtain in the future the corresponding eigenfunctions for the other sectors. This programme is very important, since for example, it may teach us to derive in a closed form the wavefunction for the interesting problem of the $\Delta = -\frac{1}{2} XXZ$ chain, or the sector $(0, n, L-n)$ ($n = 0, \dots, L$) of the $SU(3)_q$ model at $q = \exp(i2\pi/3)$.

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Appendix A. $SU(4)$ model with $q = \exp(3i\pi/4)$, sector $(1, 2, 2, L-5)$

The nested Bethe ansatz for the $SU(4)$ Perk–Schultz model with free boundary can be written as follows (see (2) and (5)):

$$\prod_{j=1, j \neq i}^{p_0} F(w_i, w_j) = \prod_{j=1}^{p_1} f(w_i, v_j) \quad i = 1, 2, \dots, p_0 \quad (89)$$

$$\prod_{j=1, j \neq i}^{p_1} F(v_i, v_j) = \prod_{j=1}^{p_0} f(v_i, w_j) \prod_{j=1}^{p_2} f(v_i, u_j) \quad i = 1, 2, \dots, p_1 \quad (90)$$

$$\prod_{j=1, j \neq i}^{p_2} F(u_i, u_j) = \prod_{j=1}^{p_1} f(u_i, v_j) \quad i = 1, 2, \dots, p_2 \quad (91)$$

$$f^L(u_i, 0) = 1 \quad i = 1, 2, \dots, p_2 \quad (92)$$

where we restricted ourselves to the solutions satisfying the FFC (7).

We have found free-fermion solutions for arbitrary values of q in the sector $(n_0, n_1, n_2, n_3) = (1, 1, 1, L - 3)$ (see section 4) and for $q = \exp(3i\pi/4)$ in the sector $(1, 1, 2, L - 4)$ (see section 5). We now intend to obtain solutions in the next simple sector $(1, 2, 2, L - 5)$. For this case $p_0 = 1, p_1 = 3, p_2 = 5$. Choosing the five variables u_i satisfying the FFC (92), we look for three variables v_i and one variable w , satisfying the nine equations given in (89)–(91).

Let us begin with the third group of equations (91). The left-hand side of these equations can be written as $L(u_i), i = 1, \dots, 5$, where

$$L(u) = \frac{1}{F(u, u)} \prod_{j=1}^5 F(u, u_j). \quad (93)$$

Using definition (5) and fixing the anisotropy parameter $\eta = 3\pi/4$ we obtain after tedious but straightforward calculations

$$\begin{aligned} \frac{L(u) - 1}{L(u) + 1} = & s(s_5 + s_3 + s_1 - c(s_4 + s_2 + 1) - c^2(s_3 + 2s_1) + c^3(s_2 + 2) \\ & + c^4 s_1 - c^5)(s_4 + s_2 + 1 - c(s_5 + s_3 + s_1) - c^2(s_4 + 2s_2 + 3) \\ & + c^3(s_3 + 2s_1) + c^4(s_2 + 3) - c^5 s_1 - c^6)^{-1} \end{aligned} \quad (94)$$

where we have used short notations for the symmetric functions

$$\begin{aligned} s_1 &= \cos 2u_1 + \cos 2u_2 + \dots + \cos 2u_5 \\ s_2 &= \cos 2u_1 \cos 2u_2 + \dots + \cos 2u_4 \cos 2u_5 \\ &\dots \\ s_5 &= \cos 2u_1 \cos 2u_2 \dots \cos 2u_5 \end{aligned} \quad (95)$$

and for the trigonometric functions: $c = \cos 2u$ and $s = \sin 2u$. It is clear that

$$\prod_{j=1}^5 (\cos 2u_j - \cos 2u) = s_5 - c s_4 + c^2 s_3 - c^3 s_2 + c^4 s_1 - c^5 = 0 \quad (96)$$

for $u = u_i$. Using the last identity we can exclude s_5 from (94). After some simplifications we obtain the more simple fraction

$$\frac{L(u) - 1}{L(u) + 1} = s \frac{s_3 + s_1 - c(s_2 + 1)}{s_4 + s_2 + 1 - c(s_3 + s_1) - c^2 + c^4} \quad u = u_1, \dots, u_5. \quad (97)$$

On the other hand, the right-hand side of the third group of equations (91) can be written as $R(u_i), i = 1, \dots, 5$, where

$$R(u) = \prod_{j=1}^3 f(u, v_j). \quad (98)$$

Again, rather straightforwardly we obtain

$$\frac{R(u) - 1}{R(u) + 1} = -s \frac{2\sqrt{2}\sigma_2 + 4c\sigma_1 + \sqrt{2}(1 + 2c^2)}{4\sigma_3 + 2\sigma_1 + 2\sqrt{2}c\sigma_2 + \sqrt{2}(3c - 2c^3)} \quad (99)$$

where we have used the short notation for the symmetric functions

$$\begin{aligned}\sigma_1 &= \cos 2v_1 + \cos 2v_2 + \cos 2v_3 \\ \sigma_2 &= \cos 2v_1 \cos 2v_2 + \cos 2v_1 \cos 2v_3 + \cos 2v_2 \cos 2v_3 \\ \sigma_3 &= \cos 2v_1 \cos 2v_2 \cos 2v_3.\end{aligned}\quad (100)$$

By equating the right-hand sides of equations (97) and (99) we obtain a polynomial equation of degree 6 with respect to the variable c . Using (96) we can exclude the terms with c^6 and c^5 and obtain a polynomial equation of the fourth degree with respect to the variable c . It is amusing to verify that all five coefficients of this polynomial become zero if we choose

$$\sigma_1 = -\frac{2s_1 + s_3 + s_1 s_2}{\sqrt{2}(1 + s_2)} \quad \sigma_2 = -\frac{1}{2} + \frac{s_1(s_1 + s_3)}{1 + s_2} \quad \sigma_3 = \frac{s_3 + 2s_5 - s_1 s_2 - 2s_1 s_4}{2\sqrt{2}(1 + s_2)}.\quad (101)$$

This result means that for arbitrary values of u_1, \dots, u_5 we calculate s_1, \dots, s_5 by using (95), and the above relations (101) together with definition (100) enable us to find the variables v_1, v_2, v_3 as the roots of the equation

$$\cos^3 2v - \sigma_1 \cos^2 2v + \sigma_2 \cos 2v - \sigma_3 = 0 \quad (v = v_1, v_2, v_3).\quad (102)$$

Let us now consider the first equation

$$\prod_{j=1}^3 f(w, v_j) - 1 \equiv R(w) - 1 = 0\quad (103)$$

where R is given by (98). Due to (99) equation (103) becomes

$$2\sqrt{2}\sigma_2 + 4 \cos 2w \sigma_1 + \sqrt{2}(1 + 2 \cos^2 2w) = 0.\quad (104)$$

Inserting (101) into this last equation we obtain

$$(s_1 - \cos 2w) \left(\frac{s_1 + s_3}{1 + s_2} - \cos 2w \right) = 0.\quad (105)$$

We have two possible solutions for $\cos 2w$. Choosing the solution

$$\cos 2w = \frac{s_1 + s_3}{1 + s_2}\quad (106)$$

we can check by using standard algebra that for this choice of w and v_1, v_2, v_3 obtained from (102) all equations of the second group of (90) are satisfied automatically.

As a result, we explained one more free-fermion branch found in our numerical observations.

Appendix B. Eigenvectors in the sector $(1, \dots, 1, L - N + 1)$

In section 4 we found a set of solutions in the sector $(1, \dots, 1, L - N + 1)$ of the $SU(N)_q$ model with an arbitrary value of the anisotropy q , and free boundary conditions. The corresponding eigenenergies are given by (8), and can be written as

$$E = \sum_{j=1}^{N-1} \epsilon_j = \sum_{j=1}^{N-1} \left(q + \frac{1}{q} - x_j - \frac{1}{x_j} \right)\quad (107)$$

where

$$x_j = \exp\left(\frac{i\pi k_j}{L}\right) \quad 1 \leq k_j \leq L - 1.\quad (108)$$

We intend to prove that the corresponding eigenvectors have the following form

$$|\psi_{\{x_1, \dots, x_{N-1}\}}\rangle = \sum_{m_1, \dots, m_{N-1}=1}^L q^{-f(m_1, \dots, m_{N-1})} \times \det \begin{vmatrix} \Psi_1(m_1) & \Psi_1(m_2) & \cdots & \Psi_1(m_{N-1}) \\ \Psi_2(m_1) & \Psi_2(m_2) & \cdots & \Psi_2(m_{N-1}) \\ \cdots & \cdots & \cdots & \cdots \\ \Psi_{N-1}(m_1) & \Psi_{N-1}(m_2) & \cdots & \Psi_{N-1}(m_{N-1}) \end{vmatrix} |m_1, \dots, m_{N-1}\rangle$$

where $|m_1, \dots, m_{N-1}\rangle$ are the vector basis representing the configuration where the i th particle is located at site m_i ($i = 1, \dots, N-1, 1 \leq m_i \leq L$). The Slater determinants entering (109) depend upon the set of one-particle amplitudes $\Psi_j(m)$, which apart from a normalization factor are given by

$$\Psi_j(m) = \left(1 - \frac{q}{x_j}\right) x_j^m - (1 - qx_j)/x_j^m. \quad (109)$$

The factor $f(m_1, \dots, m_{N-1})$ is an integer number that depends on the relative order of the sequence $\{m_1, m_2, \dots, m_{N-1}\}$. We are going to find this dependence in the process of proving (109).

The application of the Hamiltonian (1) to the above vector leads to a set of equations for the amplitudes $\Psi(m_1, \dots, m_{N-1})$.

Consider initially the amplitudes where $|m_i - m_j| \geq 2$ for all pairs (i, j) . These amplitudes give us a set of equations ('regular' ones) for one-particle amplitudes that are satisfied for every product in the determinant separately. Consider now the case where $m_i = m$ and $m_j = m + 1$ for precisely a single pair ($i \leq j$). The equation in this case is given by

$$E\Psi(\dots, m, \dots, m+1, \dots) = - \sum_{k=1, k \neq (i, j)}^{N-1} [\Psi(\dots, m_k - 1, \dots) + \Psi(\dots, m_k + 1, \dots)] \\ - \Psi(\dots, m_1, \dots, m+1, \dots) - \Psi(\dots, m, \dots, m+2) \\ - \Psi(\dots, m+1, \dots, m, \dots) \\ + [(N-1)q + (N-2)/q]\Psi(\dots, m, \dots, m+1, \dots). \quad (110)$$

Comparing this equation with the 'regular' ones, derived previously, we obtain

$$\Psi(\dots, m, \dots, m, \dots) + \Psi(\dots, m+1, \dots, m+1, \dots) - \Psi(\dots, m+1, \dots, m, \dots) \\ - \frac{1}{q}\Psi(\dots, m, \dots, m+1, \dots) = 0. \quad (111)$$

Inserting (109) into (111) we verify that the first two terms vanish while the third and the fourth ones correspond to the same determinant, apart from a minus sign. As a result, we obtain an equation for the factor f in (109):

$$f(\dots, m, \dots, m+1, \dots) = -1 + f(\dots, m+1, \dots, m, \dots). \quad (112)$$

We are now ready to fix the function f . Let $f(m_1, m_2, \dots, m_{N-1}) = 0$, for all $\{m_j\}$ satisfying to the order

$$m_1 < m_2 < \cdots < m_{N-1}. \quad (113)$$

Then (112) fixes f for all remaining configurations.

Consider, for example, the $SU(4)_q$ case. We have six possibilities

$$\begin{aligned} f(m_1 < m_2 < m_3) &= 0 \\ f(m_2 < m_1 < m_3) &= f(m_1 < m_3 < m_2) = 1 \\ f(m_2 < m_3 < m_1) &= f(m_3 < m_1 < m_2) = 2 \\ f(m_3 < m_2 < m_1) &= 3. \end{aligned}$$

It is clear that f is equal to the minimal number of pair transpositions necessary to put the configuration in the order $m_1 < m_2 < m_3$.

One can easily check that the remaining equations coming from the other amplitudes reduce to the already obtained relation (112), concluding the proof that (109) are the eigenfunctions corresponding to the NBAE solutions derived in section 4, with eigenvalues given by (107).

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